J. Math. Anal. Appl. 370 (2010) 339–349

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

An almost periodic noncommutative Wiener's Lemma ☆

Radu Balan [a], Ilya Krishtal [b,*]

[a] Department of Mathematics, CSCAMM, and Norbert Wiener Center, University of Maryland, College Park, MD, United States
[b] Department of Mathematical Sciences, Northern Illinois University, DeKalb, IL, United States

ARTICLE INFO

Article history:
Received 23 May 2009
Submitted by T. Ransford

Keywords:
Wiener's Lemma
Banach algebras
Almost periodicity
Time-frequency shifts

ABSTRACT

We develop a theory of almost periodic elements in Banach algebras and present an abstract version of a noncommutative Wiener's Lemma. The theory can be used, for example, to derive some of the recently obtained results in time-frequency analysis such as the spectral properties of the finite linear combinations of time-frequency shifts.

1. Introduction

Wiener's Tauberian Lemma [44] is a classical result in harmonic analysis which states that if a periodic function f has an absolutely convergent Fourier series and never vanishes then the function $1/f$ also has an absolutely convergent Fourier series. This result has many extensions which appear, for example, in [4,5,8–10,15,16,19,20,25,27,30,34,35,38–42]. These extensions have found applications in frame theory [2,6,15,18,26,36], time-frequency analysis [20,21,26], sampling theory [1,2,37], pseudo-differential operators [22,24], finite-section method [23,32], etc. The above list is by no means exhaustive, it represents only the tip of the iceberg.

One of the key results of this paper is another Wiener's Lemma extension which is more general than most of the cited above. We also obtain important spectral properties of operators with rationally independent Bohr spectrum and apply the results to answer certain questions motivated by the Heil–Ramanathan–Topiwala (HRT) conjecture [17]. Some of the crucial techniques we use were developed in [9,10,12].

The paper is organized as follows. In the next section we introduce almost periodicity in Banach algebras and define Fourier series with respect to a representation. In Section 3 we derive the corresponding extension of the Wiener's Lemma. We use the developed technique to present a few interesting spectral properties of elements with certain special types of Fourier series in Section 4. Finally, in Section 5 we obtain some properties of a C^*-algebra generated by time-frequency shifts which contribute to our understanding of the HRT conjecture.

2. Almost periodic Fourier series in Banach algebras

We begin with a brief introduction of almost periodicity in unital Banach algebras. The proofs of the abstract statements in this section may be found in [7,11,28,31]. For the relevant theory of Fourier series of linear operators (which is a special case) we cite [9,13].

☆ The research was supported in part by NSF grants DMS-0807896 (first author) and DMS-0908239 (second author).
* Corresponding author.
E-mail addresses: rvbalan@math.umd.edu (R. Balan), krishtal@math.niu.edu (I. Krishtal).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.04.053

Let $\mathcal{B}$ be a unital Banach algebra with the unit element denoted by I. As usually, the main example is the algebra $End\,X$ of all bounded linear operators (endomorphisms) of a (complex) Banach space X. Let also $\mathbb{G}$ be a locally compact Abelian (LCA-)group and $\hat{\mathbb{G}}$ be its Pontryagin dual (with the algebraic operation written additively on both). By $\mathbb{G}_d$ we denote the group $\mathbb{G}$ endowed with discrete topology and by $\hat{\mathbb{G}}_c$ its dual – the Bohr compactification of $\hat{\mathbb{G}}$.

A continuous function $\varphi : \mathbb{G} \to \mathcal{B}$ is *Bohr almost periodic* if, for every $\varepsilon > 0$, the set $\Omega(\varepsilon) = \{\omega \in \mathbb{G}\colon \sup_{g\in\mathbb{G}} \|\varphi(g+\omega) - \varphi(g)\| < \varepsilon\}$ of its ε-periods is relatively dense in $\mathbb{G}$, i.e., there exists a compact set $K = K_\varepsilon \subset \mathbb{G}$ such that $(g+K)\cap\Omega(\varepsilon) \neq \emptyset$ for all $g \in \mathbb{G}$.

Let $T : \hat{\mathbb{G}} \to End\,\mathcal{B}$ be an isometric representation of the group $\hat{\mathbb{G}}$ with the following properties:

- $T(\gamma)I = I$ for all $\gamma \in \hat{\mathbb{G}}$;
- $T(\gamma)(AB) = (T(\gamma)A)(T(\gamma)B)$ for all $\gamma \in \hat{\mathbb{G}}$, $A, B \in \mathcal{B}$.

We refer the reader to Section 5 and [12] for a variety of examples of representations with the above property.

Definition 2.1. We say that $A \in \mathcal{B}$ is *T-almost periodic*, or $A \in AP^T(\mathcal{B})$, if the function $\hat{A} : \hat{\mathbb{G}} \to \mathcal{B}$, $\hat{A}(\gamma) = T(\gamma)A$, is continuous (in the topology of $\mathcal{B}$) and Bohr almost periodic.

It is known, that for $A \in AP^T(\mathcal{B})$ the orbit $\{T(\gamma)A,\ \gamma \in \hat{\mathbb{G}}\}$ is totally bounded (precompact) and the function $\hat{A}$ has a unique continuous extension to the Bohr compactification $\hat{\mathbb{G}}_c$. We denote this extension by the same symbol $\hat{A}$. Consider the Fourier series of the function $\hat{A}$

$$\hat{A}(\gamma) \sim \sum_{i\in\mathbb{Z}} \gamma(g_i)A_i, \quad g_i \in \mathbb{G}, \tag{2.1}$$

where the elements $A_i \in \mathcal{B}$ are eigen-vectors of the representation T, that is $T(\gamma)A_i = \gamma(g_i)A_i$, $\gamma \in \hat{\mathbb{G}}$. The series in (2.1) will be called the *Fourier series* of A and the elements A_i – the *Fourier coefficients* of A. The set $\{g_i\}$ of the group elements in (2.1) will be referred to as the *Bohr spectrum* of the element A and denoted by $\Lambda(A)$. Instead, the set $\sigma(A)$ denotes the usual *spectrum* of A with respect to the algebra $\mathcal{B}$. Thus, $\lambda I - A$ is invertible in $\mathcal{B}$ for all $\lambda \in \rho(A) = \mathbb{C}\setminus\sigma(A)$, the resolvent set of A.

The coefficients A_i can be obtained via

$$A_i = \int_{\hat{\mathbb{G}}_c} \hat{A}(\gamma)\gamma(-g_i)\,\bar{\mu}(d\gamma),$$

where $\bar{\mu}$ is the normalized Haar measure on $\hat{\mathbb{G}}_c$. They may also be computed using the notion of g-nets, see [12, §4] and references therein.

Definition 2.2. Let Ω be a filtered set. A bounded net of functions $f_\alpha \in L^1(\hat{\mathbb{G}})$, $\alpha \in \Omega$, is called a *g-net* for some $g \in \mathbb{G}$ if the following two conditions are satisfied:

- $\hat{f}_\alpha(g) = 1$ for all $\alpha \in \Omega$;
- $\lim_\alpha f_\alpha * f = 0$ for every $f \in L^1(\hat{\mathbb{G}})$ such that $\hat{f}(g) = 0$.

Example 2.1. The simplest example of a 0-net in $L^1(\mathbb{R}^d)$ is given by the following family of step functions:

$$f_N(x) = \begin{cases} \frac{1}{(2N)^d}, & x \in [-N,N]^d; \\ 0, & x \notin [-N,N]^d. \end{cases} \tag{2.2}$$

We cite [12, Remark 4.5] for an example of a compactly supported g-net in $L^1(\hat{\mathbb{G}})$ for a general LCA-group $\mathbb{G}$.

If (f_α) is a g_i-net in $L^1(\hat{\mathbb{G}})$ and μ is the Haar measure on $\hat{\mathbb{G}}$, the following formula can be used to find the Fourier coefficient A_i of an element $A \in AP^T(\mathcal{B})$:

$$A_i = \lim_\alpha \int_{\hat{\mathbb{G}}} f_\alpha(\gamma)\hat{A}(-\gamma)\,\mu(d\gamma). \tag{2.3}$$

The Fourier coefficient is independent of the choice of a particular g_i-net.

Remark 2.1. Observe that if $A \in AP^T(\mathcal{B})$ is invertible in $\mathcal{B}$ then $A^{-1} \in AP^T(\mathcal{B})$. This follows from the property of precompactness of the orbit and the inequalities

$$\begin{aligned}\|\widehat{A^{-1}}(\gamma) - \widehat{A^{-1}}(\tau)\| &= \|\widehat{A^{-1}}(\gamma)\big(\hat{A}(\tau) - \hat{A}(\gamma)\big)\widehat{A^{-1}}(\tau)\| \\ &\leqslant \|A^{-1}\|^2 \|\hat{A}(\tau) - \hat{A}(\gamma)\|.\end{aligned}$$

Moreover, it can be shown [31] that the Bohr spectrum of the inverse element A^{-1} is contained in the smallest subgroup of $\mathbb{G}$ generated by $\Lambda(A)$.

3. Wiener's Lemma in AP

As usual when Wiener's Lemma is discussed, we are interested in elements $A \in \mathcal{B}$ whose Fourier series are summable or summable with a weight.

Definition 3.1. A *weight* is a function $\nu : \mathbb{G} \to [1, \infty)$ such that

$$\nu(g_1 + g_2) \leqslant \nu(g_1)\nu(g_2), \quad \text{for all } g_1, g_2 \in \mathbb{G}.$$

A weight is *admissible* if it satisfies the *GRS-condition*

$$\lim_{n\to\infty} n^{-1} \ln \nu(ng) = 0, \quad \text{for all } g \in \mathbb{G}.$$

For a weight ν, by $AP^T_\nu(\mathcal{B})$ we will denote the subset of $AP^T(\mathcal{B})$ of elements with ν-absolutely convergent Fourier series, i.e.

$$\|A\|_\nu = \sum_{i\in\mathbb{Z}} \nu(g_i)\|A_i\| < \infty.$$

Lemma 3.1. *The set $AP^T_\nu(\mathcal{B})$ is a Banach algebra with respect to the norm $\|\cdot\|_\nu$.*

Proof. The proof is the same as in Lemma 2 in [9] or [10]. In particular, $AP^T_\nu(\mathcal{B})$ is obviously a closed subspace of $\ell^1_\nu(\mathbb{G}_d, \mathcal{B})$ and, hence, is itself a Banach space. Moreover, it is easily verified that $\ell^1_\nu(\mathbb{G}_d, \mathcal{B})$ is an algebra with respect to the discrete convolution and $AP^T_\nu(\mathcal{B})$ is its subalgebra. The Banach algebra property

$$\|AB\|_\nu \leqslant \|A\|_\nu \|B\|_\nu$$

follows from the submultiplicativity of the weight. Observe that the admissibility condition is not required for this result. □

The following is the main result of this section.

Theorem 3.2. *Let ν be an admissible weight. Then the subalgebra $AP^T_\nu(\mathcal{B}) \subset \mathcal{B}$ is inverse closed, that is, if $A \in AP^T_\nu(\mathcal{B})$ is invertible in $\mathcal{B}$ then $A^{-1} \in AP^T_\nu(\mathcal{B})$.*

Careful examination of the proof of a less general result in [9, §2] shows that it extends almost without change to the setting studied in this paper. We will present a sketch of the proof for completeness. Before doing so, however, we observe that by considering the left regular representation of the algebra of almost periodic functions we obtain the following classical version of the theorem.

Corollary 3.3. *(See [30].) Let $f \in L^\infty(\mathbb{G})$ be an almost periodic function that never vanishes and has summable Fourier coefficients. Then the (almost periodic) function $1/f$ also has summable Fourier coefficients.*

Let us now begin the sketch of the proof of Theorem 3.2. We start by quoting the celebrated Bochner–Phillips theorem [14]. Let $\mathfrak{B}$ be a unital Banach algebra with the following properties.

- There exist a closed subalgebra $\mathcal{F} \subset \mathfrak{B}$ and a closed commutative subalgebra $\mathcal{A}$ from the center of $\mathfrak{B}$ such that the elements

  $$(a, f) = \sum_{k=1}^n a_k f_k, \quad a = (a_1, \dots, a_n) \in \mathcal{A}^n, \ f = (f_1, \dots, f_n) \in \mathcal{F}^n,$$

  are dense in $\mathfrak{B}$.

- $\|a_0 f_0\| = \|a_0\| \|f_0\|$ for all $a_0 \in \mathcal{A}$, $f_0 \in \mathcal{F}$.
- $\|\sum_{k=1}^{n} \chi(a_i) f_i\| \leqslant \|(a, f)\|$ for all $a = (a_1, \dots, a_n) \in \mathcal{A}^n$, $f = (f_1, \dots, f_n) \in \mathcal{F}^n$ and any character (complex algebra homomorphism) χ from the spectrum $Sp\,\mathcal{A}$ of the algebra $\mathcal{A}$.

An algebra homomorphism $\bar{\chi} : \mathfrak{B} \to \mathcal{F}$ is called a *generalized character* if there exists a complex character $\chi \in Sp\,\mathcal{A}$ such that $\bar{\chi}(af) = \chi(a) f$ for all $a \in \mathcal{A}$, $f \in \mathcal{F}$. The set of all generalized characters will be denoted by $Sp(\mathfrak{B}, \mathcal{F})$.

Theorem 3.4 *(Bochner–Phillips). An element $b \in \mathfrak{B}$ has a left (right) inverse if and only if for every generalized character $\bar{\chi} \in Sp(\mathfrak{B}, \mathcal{F})$ the element $\bar{\chi}(b) \in \mathcal{F}$ has a left (right) inverse in $\mathcal{F}$.*

We need the following special case of the above theorem. Let $\mathfrak{B} = L_\nu(\mathbb{G}_d, \mathcal{B})$ be the algebra of $\mathcal{B}$-valued functions on $\mathbb{G}_d$ summable with the weight ν with the algebraic operation given by (discrete) convolution and the unit element denoted by δ_0. Then the subalgebras $\mathcal{A} = \{fI,\ f \in L_\nu(\mathbb{G}_d) = L_\nu(\mathbb{G}_d, \mathbb{C})\}$ and $\mathcal{F} = \{A\delta_0\colon\ A \in \mathcal{B}\}$ are easily seen to satisfy the conditions of Theorem 3.4. Moreover, since ν is an admissible weight, all generalized characters in $Sp(\mathfrak{B}, \mathcal{F})$ are determined by the Fourier transform on $\mathfrak{B}$. Hence, we have the following result.

Corollary 3.5. *An element $f \in L_\nu(\mathbb{G}_d, \mathcal{B})$ is invertible in $L_\nu(\mathbb{G}_d, \mathcal{B})$ if and only if all elements in $\mathcal{B}$ of the form*

$$\hat{f}(\gamma) = \sum_{g \in \mathbb{G}_d} f(g)\gamma(-g), \quad \gamma \in \hat{\mathbb{G}}_c,$$

are invertible in $\mathcal{B}$.

We are now ready to complete the proof of Theorem 3.2.

Proof. Let $A \in AP_\nu^T(\mathcal{B})$ be invertible in $\mathcal{B}$. Consider the function $f : \mathbb{G}_d \to \mathcal{B}$ defined by $f(g_i) = A_i$, $g_i \in \mathbb{G}_d$, where A_i are the Fourier coefficients of A (we set $f(g) = 0$ if $g \notin \Lambda(A)$). Clearly, $f \in L_\nu(\mathbb{G}_d, \mathcal{B})$ and $\hat{f}(\gamma) = \hat{A}(-\gamma)$. Moreover, these operators are invertible:

$$\hat{f}(\gamma)^{-1} = \big(\hat{A}(-\gamma)\big)^{-1} = \big(T(-\gamma)A\big)^{-1} = T(-\gamma)\big(A^{-1}\big), \quad \gamma \in \hat{\mathbb{G}}_c.$$

Hence, by Corollary 3.5, f is invertible in $L_\nu(\mathbb{G}_d, \mathcal{B})$ and for $B = A^{-1}$ we have

$$\hat{B}(\gamma) = T(\gamma)B = \sum_{g \in \mathbb{G}_d} f^{-1}(g)\gamma(-g).$$

Therefore, $B = A^{-1} \in AP_\nu^T(\mathcal{B})$. $\square$

Suppose now that $\mathbb{G} = \mathbb{R}^d \simeq \hat{\mathbb{G}}$, $\hat{A}(x) = \sum_{j \in \mathbb{R}^d} e^{2\pi i \langle x, j\rangle} A_j \in \mathcal{B}$, and the Fourier coefficients of A satisfy

$$\|A\|_{\nu_\rho} = \sum_{j \in \mathbb{R}^d} e^{\rho|j|} \|A_j\| < \infty \tag{3.1}$$

for some $\rho > 0$. Note, the exponential weight $\nu_\rho(j) = e^{\rho|j|}$, $j \in \mathbb{R}^d$, is submultiplicative but does not satisfy the GRS condition from Definition 3.1, and, hence, is not admissible. The conclusion of Theorem 3.2 fails for $AP_{\nu_\rho}^T(\mathcal{B})$, however, it is possible to prove a slightly weaker result. Observe that the algebra $AP_{\nu_\rho}^T(\mathcal{B})$ is different from the algebra of operators with exponential decay of Fourier coefficients considered in [9,10], which is, essentially, $\bigcup_{\rho>0} AP_{\nu_\rho}^T(\mathcal{B})$. The latter algebra is not a Banach algebra since it is not complete. Also, if $\Lambda(A)$ is bounded, the Fourier coefficients of $A \in AP_{\nu_\rho}^T(\mathcal{B})$ do not necessarily have exponential decay.

Theorem 3.6. *Let $\nu_\rho(j) = e^{\rho|j|}$, $j \in \mathbb{R}^d$, be an exponential weight and $A \in AP_{\nu_\rho}^T(\mathcal{B})$ be invertible in $\mathcal{B}$. Then there exists $\bar{\rho} > 0$ such that $A^{-1} \in AP_{\nu_{\bar{\rho}}}^T(\mathcal{B})$.*

Proof. In this case, the function $\hat{A}$ extends holomorphically to the interior of the closed band

$$\mathbb{C}_\rho^d = \big\{z = x + iy\colon\ x, y \in \mathbb{R}^d,\ |y| \leqslant \rho\big\}.$$

We call this extension $\bar{A}$,

$$\bar{A}(z) = \sum_{j \in \mathbb{R}^d} e^{2\pi i \langle z, j\rangle} A_j.$$

Note $\bar{A}(x) = \hat{A}(x)$, for all $x \in \mathbb{R}^d$.

We assume that $A \in AP^T_{\nu_\rho}(\mathcal{B})$ is invertible in $\mathcal{B}$. Hence, $\hat{A}(x)$ is invertible in $\mathcal{B}$ for all $x \in \mathbb{R}^d$. Moreover, since

$$\bar{A}(x+iy) = \hat{A}(x) + \bar{A}(x+iy) - \hat{A}(x) = \hat{A}(x)\big(I + \big(\hat{A}(x)\big)^{-1}\big(\bar{A}(x+iy) - \hat{A}(x)\big)\big),$$

$\bar{A}(x+iy)$ is invertible in $\mathcal{B}$ as soon as $\|\bar{A}(x+iy) - \hat{A}(x)\|_{\mathcal{B}} < \|\hat{A}(x)^{-1}\|^{-1}_{\mathcal{B}} = \|A^{-1}\|^{-1}_{\mathcal{B}}$. Let $|y| \leqslant \bar{\rho}$ for some $\bar{\rho} > 0$. Then

$$\begin{aligned}\big\|\bar{A}(x+iy) - \hat{A}(x)\big\|_{\mathcal{B}} &= \Big\|\sum_{j\in\mathbb{R}^d} e^{2\pi i\langle x+iy,j\rangle} A_k - \sum_{j\in\mathbb{R}^d} e^{2\pi i\langle x,j\rangle} A_j\Big\| \\ &\leqslant \sum_{j\in\mathbb{R}^d} \big|e^{2\pi i\langle x+iy,j\rangle} - e^{2\pi i\langle x,j\rangle}\big|\|A_j\| \leqslant \sum_{j\in\mathbb{R}^d} \big|e^{-2\pi\langle y,j\rangle} - 1\big|\|A_j\|.\end{aligned}$$

For a compact subset $K \subset \mathbb{R}^d$ let

$$A^{(K)} = \sum_{j\in K} A_j \quad \text{and} \quad A^{(\bar{K})} = A - A^{(K)} = \sum_{j\in\mathbb{R}^d\setminus K} A_j.$$

Since $A \in AP^T_{\nu_\rho}(\mathcal{B}) \subset AP^T_1(\mathcal{B})$, we can choose K so that $\|A^{(\bar{K})}\|_1 \leqslant \|A^{(\bar{K})}\|_{\nu_\rho} < \frac{1}{4}\|A^{-1}\|^{-1}$. Then from the above inequalities for $\bar{\rho} \leqslant \frac{\rho}{2\pi}$ we get

$$\begin{aligned}\big\|\bar{A}(x+iy) - \hat{A}(x)\big\|_{\mathcal{B}} &\leqslant \sum_{j\in K} \big|e^{-2\pi\langle y,j\rangle} - 1\big|\|A_j\| + \sum_{j\in\mathbb{R}^d\setminus K} \big|e^{-2\pi\langle y,j\rangle} - 1\big|\|A_j\| \\ &\leqslant \sup_{j\in K}\big|e^{-2\pi\bar{\rho}|j|} - 1\big|\big\|A^{(K)}\big\|_1 + 2\big\|A^{(\bar{K})}\big\|_{\nu_\rho}.\end{aligned}$$

The above quantity is clearly less than $\|A^{-1}\|^{-1}$ for sufficiently small $\bar{\rho}$ which depends only on $\|A\|_{\nu_\rho}$, $\|A^{-1}\|_{\mathcal{B}}$, and $\Lambda(A)$.

Hence, if $B = A^{-1}$, the function $\hat{B}$ admits a bounded holomorphic extension

$$\bar{B}(z) = \big(\bar{A}(z)\big)^{-1} = \sum_{j\in\mathbb{R}^d} e^{2\pi i\langle z,j\rangle} B_j \in AP^T_1(\mathcal{B}), \quad z \in \mathbb{C}^d_{\bar{\rho}}. \tag{3.2}$$

To see that $\bar{B}(z)$ indeed has the above series representation, observe, first, that $\bar{B}(z) \in AP^T_1(\mathcal{B})$ for all $z \in \mathbb{C}^d_{\bar{\rho}}$ due to Theorem 3.2. Hence, for every $j \in \mathbb{R}^d$ and $z \in \mathbb{C}^d_{\bar{\rho}}$ there exists the Fourier coefficient

$$B_j(z) = \lim_{N\to\infty} \frac{1}{(2N)^d} \int_{[-N,N]^d} e^{-2\pi i\langle t,j\rangle} T(-t)\bar{B}(z)\,dt.$$

Moreover, since convergence of the above limit and integral is absolute and uniform in z, the functions $B_j(z)$ are holomorphic in $\mathbb{C}^d_{\bar{\rho}}$. Finally, since $B_j(x) = e^{2\pi i\langle x,j\rangle} B_j$, where B_j, $j \in \mathbb{R}^d$, are the Fourier coefficients of B, and the holomorphic extension is unique, we get the series representation in (3.2). This representation clearly implies that $A^{-1} \in AP^T_{\nu_{\bar{\rho}}}(\mathcal{B})$. □

Since $\bar{\rho}$ in the above theorem depends only on $\|A\|_{\nu_\rho}$, $\|A^{-1}\|_{\mathcal{B}}$, and $\Lambda(A)$, we have the following slightly stronger result. We let $dist(g, S) = \inf\{|g - x|,\ x \in S\}$ denote the distance between g and a set S.

Theorem 3.7. *Let $\nu_\rho(j) = e^{\rho|j|}$ be an exponential weight. Then for every $A \in AP^T_{\nu_\rho}(\mathcal{B})$ and $\varepsilon > 0$ there exists $\bar{\rho} > 0$ such that $(\lambda I - A)^{-1} \in AP^T_{\nu_{\bar{\rho}}}(\mathcal{B})$ for every $\lambda \in \mathbb{C}$ such that $dist(\lambda, \sigma(A)) \geqslant \varepsilon$.*

In the case when the spectrum of A admits a disjoint decomposition $\sigma(A) = S_1 \cup S_2$ into two nonempty separated components (this means that there exists a closed Jordan curve contained in the resolvent set that separates S_1 from S_2) then holomorphic functional calculus (see [33]) can be used to show the following.

Corollary 3.8. *Let $\nu_\rho(j) = e^{\rho|j|}$ be an exponential weight, and $A \in AP^T_{\nu_\rho}(\mathcal{B})$ be such that its spectrum (in $\mathcal{B}$) admits a decomposition $\sigma(A) = S_1 \cup S_2$ into two nonempty separated components. Let $\gamma : [0, 1] \to \rho(A)$ be a closed Jordan curve separating S_1 from S_2. Then*

$$P = \frac{1}{2\pi i}\int_\gamma (zI - A)^{-1}\,dz \tag{3.3}$$

is a non-trivial idempotent in $AP^T_{\nu_{\bar{\rho}}}(\mathcal{B})$ for some $\bar{\rho} > 0$, that is, $P^2 = P$ and $P \in AP^T_{\nu_{\bar{\rho}}}(\mathcal{B})\setminus\{0, I\}$. Furthermore, when $\mathcal{B}$ is a C^*-algebra then $P^* = P$.

4. Some spectral properties of AP elements

In [13] there is a detailed study of the spectral properties of elements with two-point Bohr spectrum. We begin this section by presenting a similar result for elements with rationally independent Bohr spectrum.

Definition 4.1. We say that the spectrum $\Lambda(A) = \{g_k;\ k \in \mathbb{N}\}$ of an element $A \in AP^T(\mathcal{B})$ is (*finitely*) *rationally independent* if

$$\sum_{k\in\Omega} r_k \cdot g_k \neq 0 \quad \text{for all (finite) } \Omega \subset \mathbb{N} \text{ and } r = (r_1, \dots, r_n, \dots) \in \mathbb{Z}^\infty \backslash \{0\}.$$

Here $r_k \cdot g_k = \sum_{i=1}^{r_k} g_k \in \mathbb{G}$ if $r_k > 0$, $r_k \cdot g_k = -((-r_k) \cdot g_k)$ if $r_k < 0$, and $0 \cdot g_k = 0 \in \mathbb{G}$.

The proposition below follows immediately from [11, Theorem 3.6.11].

Proposition 4.1. *Assume that $A \in AP^T(\mathcal{B})$ has finitely rationally independent Bohr spectrum. Then the Fourier series of A converges unconditionally to A with the constant of unconditional convergence equal to one.*

The following theorem is the key result of the section.

Theorem 4.2. *Assume that $A \in AP^T(\mathcal{B})$ has finite rationally independent Bohr spectrum $\Lambda(A) = \{g_1, g_2, \dots, g_n\}$, $n \in \mathbb{N}$. Then the spectrum $\sigma(A)$ in the Banach algebra $\mathcal{B}$ (or $AP_1^T(\mathcal{B})$) is invariant under rotations around the origin in $\mathbb{C}$.*

Proof. Let $\mathbb{G}_k$ be the smallest subgroup of $\mathbb{G}_d$ that contains $g_k \in \Lambda(A)$, $\mathbb{G}_A = \bigoplus_{k=1}^n \mathbb{G}_k$, and $\mathcal{A}_A = \{B \in AP_1^T(\mathcal{B}):\ \Lambda(B) \in \mathbb{G}_A\}$. Obviously, $\mathbb{G}_A$ is the smallest subgroup of $\mathbb{G}_d$ that contains $\Lambda(A)$ and $\mathcal{A}_A$ is inverse closed by Remark 2.1. Observe also that, because of rational independence, a general element $B \in \mathcal{A}_A$ has a unique representation of the form

$$B = \sum_{g\in\mathbb{G}_A} B_g = \sum_{k\in\mathbb{Z}^n} B_{k\cdot\Lambda(A)},$$

where $k \cdot \Lambda(A) = \sum_{i=1}^n k_i \cdot g_i$. Hence, we can define a representation $T_A : \mathbb{T}^n \to End\,\mathcal{A}_A$ by

$$T_A(\theta)B = \sum_{k\in\mathbb{Z}^n} \theta^k B_{k\cdot\Lambda(A)},$$

where $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{T}^n$ and $\theta^k = \theta_1^{k_1} \cdot \dots \cdot \theta_n^{k_n}$. It is easily verified that this representation satisfies the assumptions preceding Definition 2.1. Hence, for every $\theta \in \mathbb{T}^n$ and $\lambda \notin \sigma(A)$

$$T_A(\theta)(A - \lambda I)^{-1} = \big(T_A(\theta)A - \lambda I\big)^{-1} \in \mathcal{A}_A \subset AP_1^T(\mathcal{B})$$

and it remains to take $\theta = (\theta_0, \dots, \theta_0)$ for all $\theta_0 \in \mathbb{T}$ to obtain

$$T_A(\theta_0, \dots, \theta_0)(A - \lambda I)^{-1} = (\theta_0 A - \lambda I)^{-1} \in \mathcal{A}_A$$

and complete the proof. □

Remark 4.1. A theorem similar to the above lies at the core of the paper [13] and leads to results on exponential dichotomy for certain abstract differential and difference equations. Clearly, Theorem 4.2 can be used in a similar way, however, we prefer to develop these results elsewhere. Certain generalizations of Theorem 4.2 for elements with rationally dependent Bohr spectrum are also possible. For related results see [3].

Using Proposition 4.1, we obtain the following stronger version of Theorem 4.2. In the proof we shall use the notation

$$R(\lambda; B) = (B - \lambda I)^{-1}$$

for the *resolvent* of the element $B \in \mathcal{B}$, $\lambda \notin \sigma(B)$.

Theorem 4.3. *Assume that $A \in AP^T(\mathcal{B})$ has finitely rationally independent Bohr spectrum $\Lambda(A) = \{g_1, g_2, \dots, g_n, \dots\}$. Then the spectrum $\sigma(A)$ in the Banach algebra $\mathcal{B}$ is invariant under rotations around the origin in $\mathbb{C}$.*

Proof. Let $A = \sum_{k\in\mathbb{Z}} A_k$, where the series converges unconditionally by Proposition 4.1, and fix $\lambda \notin \sigma(A)$. For $m \in \mathbb{N}$ we define $A_{(m)}$ and $D_{(m)}$ via

$$A_{(m)} = \sum_{|k|\leqslant m} A_k \quad \text{and} \quad D_{(m)} = A - A_{(m)}.$$

Below we assume that $m \in \mathbb{N}$ is big enough so that $\|D_{(m)}\| \leqslant \frac{1}{2}\|R(\lambda; A)\|^{-1}$. Then we have

$$\begin{aligned}\big\|R(\lambda; A_{(m)})\big\| &\leqslant \big\|R(\lambda; A)\big\| \cdot \big\|\big(I - R(\lambda; A)D_{(m)}\big)^{-1}\big\| \\ &\leqslant \big\|R(\lambda; A)\big\| \sum_{k=0}^{\infty}\big\|R(\lambda; A)\big\|^k \|D_{(m)}\|^k \leqslant 2\big\|R(\lambda; A)\big\|.\end{aligned}$$

Since $\Lambda(A_{(m)})$ is finite for any $m \in \mathbb{N}$, we can use Theorem 4.2 together with Proposition 4.1 to obtain

$$\big\|R(\theta\lambda; A_{(m)})\big\| \leqslant 2\big\|R(\lambda; A)\big\|, \quad \text{for all } \theta \in \mathbb{T}.$$

Using the above inequality, we get for big $m, n \in \mathbb{N}$

$$\begin{aligned}\big\|R(\theta\lambda; A_{(m)}) - R(\theta\lambda; A_{(n)})\big\| &\leqslant \big\|R(\theta\lambda; A_{(m)})\big\| \cdot \big\|R(\theta\lambda; A_{(n)})\big\| \cdot \|D_{(m)} - D_{(n)}\| \\ &\leqslant 4\big\|R(\lambda; A)\big\|^2\big(\|D_{(m)}\| + \|D_{(n)}\|\big), \quad \theta \in \mathbb{T}.\end{aligned}$$

Hence, the sequence $\{R(\theta\lambda; A_{(m)})\}_{m\in\mathbb{N}}$ is Cauchy for every $\theta \in \mathbb{T}$ and, therefore, converges to $R(\theta\lambda; A)$. Thus, $\theta\lambda \notin \sigma(A)$ and the theorem is proved. $\square$

The following theorem presents a class of elements that cannot be idempotent. We shall use it in the next section to derive some spectral properties of so-called causal operators.

Theorem 4.4. *Let $A \in AP^T(\mathcal{B})$ and assume that there exists $\lambda \in \Lambda(A)$ such that $\lambda \neq \lambda_1 + \lambda_2$ for all $\lambda_1, \lambda_2 \in \Lambda(A)$. Then $A^2 \neq A$.*

Proof. Let $A \in AP^T(\mathcal{B})$ and $\lambda \in \Lambda(A)$ have the above property. It is immediate (see also [12, Corollary 7.8]), that

$$\Lambda(MN) \subset \Lambda(M) + \Lambda(N) \quad \text{for all } M, N \in AP^T(\mathcal{B}). \tag{4.1}$$

Hence, $\lambda \notin \Lambda(A^2)$ and, therefore, $A^2 \neq A$. $\square$

5. Time-frequency shifts and the HRT conjecture

Here we illustrate the significance of the above results in time-frequency analysis and their connection with the HRT conjecture. In this section the algebra $\mathcal{B}$ is assumed to be $End\, L^p(\mathbb{G})$, $p \in [1, \infty)$.

The standard examples of representations are typically provided by translations

$$S : \mathbb{G} \to \mathcal{B}, \qquad \big(S(g)f\big)(x) = f(x - g), \quad x, g \in \mathbb{G}, \ f \in L^p(\mathbb{G}),$$

and modulations

$$M : \hat{\mathbb{G}} \to \mathcal{B}, \qquad \big(M(\gamma)f\big)(x) = \langle\gamma, x\rangle f(x), \quad x \in \mathbb{G}, \ \gamma \in \hat{\mathbb{G}}, \ f \in L^p(\mathbb{G}).$$

The representation T is then assumed to be either

$$T : \mathbb{G} \to End\,\mathcal{B}, \qquad T(g)A = S(g)AS(-g), \quad A \in \mathcal{B},$$

or

$$T : \hat{\mathbb{G}} \to End\,\mathcal{B}, \qquad T(\gamma)A = M(\gamma)AM(-\gamma), \quad A \in \mathcal{B}.$$

We, however, are more interested in the time-frequency analysis. For this reason, we consider an LCA-group $\mathbb{G} \times \hat{\mathbb{G}}$ and a Weyl representation $T : \hat{\mathbb{G}} \times \mathbb{G} \to End\,\mathcal{B}$ defined by

$$T(\gamma, g)A = S(g)M(\gamma)AM(-\gamma)S(-g), \quad g \in \mathbb{G}, \ \gamma \in \hat{\mathbb{G}}, \ A \in \mathcal{B}. \tag{5.1}$$

It is immediate that this is a representation that satisfies the assumptions preceding Definition 2.1. For brevity, we will denote $U_\lambda = U_{g,\gamma} = M(\gamma)S(g)$ and refer to these operators as *time-frequency shifts*. Below we shall always assume the following.

Assumption 5.1. The group $\mathbb{G}$ is such that any eigen-vector of T is a constant multiple of some U_λ.

Observe that the group $\mathbb{R}^d$ naturally satisfies the above assumption because the only bounded linear operators on $L^p(\mathbb{R}^d)$, $p \in [1,\infty)$, that commute with all translations and modulations are scalar multiples of the identity.

From the results in Section 3, we immediately get the following two corollaries (see [5] for analogous results).

Corollary 5.2. *The algebra of all ν-summable time-frequency shifts coincides with $AP^T_\nu(\mathcal{B})$ and, therefore, is inverse closed.*

Corollary 5.3. $\nu(\lambda) = \nu_\rho(\lambda) = e^{\rho|\lambda|}$ *be an exponential weight, $\mathbb{G} = \mathbb{R}^d$ and $A \in AP^T_{\nu_\rho}(\mathcal{B})$ be an invertible operator. Then there exists $\bar\rho > 0$ such that $A^{-1} \in AP^T_{\nu_{\bar\rho}}(\mathcal{B})$.*

In the case $p = 2$, $\mathcal{B}$ is a C^*-algebra and Corollary 3.8 implies the following result (for the definition of the pseudoinverse see [33]).

Corollary 5.4. *Let $\nu(\lambda) = \nu_\rho(\lambda) = e^{\rho|\lambda|}$ be an exponential weight, $\mathbb{G} = \mathbb{R}^d$ and $A \in AP^T_{\nu_\rho}(End\, L^2(\mathbb{G}))$ be an operator with closed range. Then both its pseudoinverse $A^\# \in AP^T_{\nu_{\bar\rho}}(End\, L^2(\mathbb{G}))$ and the orthogonal projection onto its range $P_{Ran\, A} \in AP^T_{\nu_{\bar\rho}}(End\, L^2(\mathbb{G}))$, for some $\bar\rho > 0$.*

Remark 5.1. Similar to [6,21,26] one can obtain localization results for canonical duals of Weyl–Heisenberg frames immediately from the above corollaries. We will explore these consequences elsewhere.

Next, we address the question of the faithful tracial state on the C^*-algebra

$$\mathfrak{U} = AP^T\big(End\big(L^2(\mathbb{G})\big)\big).$$

It is known (see, e.g., [5]) that $\Gamma : \mathfrak{U} \to \mathbb{C}$, $\Gamma(\sum_\lambda c_\lambda U_\lambda) = c_0$ defines such a state. We, however, can give a more explicit formula using (2.3). As an immediate consequence of [12, Theorem 4.19(i)] we obtain

$$\begin{aligned}\tau(A) &= \int\limits_{(\hat{\mathbb{G}}\times\mathbb{G})_c} \hat{A}(-\gamma,-g)\,\bar\mu\big(d(\gamma,g)\big)\\ &= \lim_\alpha \int\limits_{\hat{\mathbb{G}}\times\mathbb{G}} f_\alpha(\gamma,g)T(-\gamma,-g)\,A\mu\big(d(g,\gamma)\big) = c_0 I,\end{aligned} \tag{5.2}$$

where the limit and the integrals converge in the uniform operator topology. Hence, the faithful tracial state Γ admits a representation

$$\begin{aligned}\Gamma(A) &= \int\limits_{(\hat{\mathbb{G}}\times\mathbb{G})_c} \big\langle \hat{A}(-\gamma,-g)x, x\big\rangle\,\bar\mu\big(d(g,\gamma)\big)\\ &= \lim_\alpha \int\limits_{\hat{\mathbb{G}}\times\mathbb{G}} f_\alpha(\gamma,g)\big\langle\big(T(-\gamma,-g)A\big)x, x\big\rangle\,\mu\big(d(g,\gamma)\big),\end{aligned}$$

where (f_α) is a 0-net in $L^1(\hat{\mathbb{G}} \times \mathbb{G})$ and $x \in L^2(\mathbb{G})$ has norm 1.

The following analog of Theorem 5.5.8 in [43] is now immediate.

Theorem 5.5. *Let $\mathbb{G}$ satisfy Assumption* 5.1*. Then the C^*-algebra $\mathfrak{U}$ contains no proper* (*closed*) *C^*-ideals.*

Proof. Indeed, if $\mathcal{I}$ is a closed C^*-ideal and $0 \neq A^*A \in \mathcal{I}$ then, obviously, $\widehat{A^*A}(-\gamma,-g) \in \mathcal{I}$ for all $(\gamma, g) \in (\hat{\mathbb{G}} \times \mathbb{G})_c$ and, therefore, $0 \neq \tau(A^*A) = c_0 I \in \mathcal{I}$ by (5.2). $\square$

Corollary 5.6. *If $\mathbb{G}$ is an infinite group* (*satisfying Assumption* 5.1)*, the algebra $\mathfrak{U}$ contains no non-trivial compact operators.*

Corollary 5.7. *If $\mathbb{G}$ is an infinite group* (*satisfying Assumption* 5.1)*, the algebra $\mathfrak{U}$ contains no non-trivial finite rank projections.*

Proof. The result is, of course, immediate since finite rank projections are compact, but we find it instructive to show that if P is a rank-one projection on $L^2(\mathbb{R}^d)$ then $\Gamma(P)=0$. Let $Px=\langle x, f\rangle f$, $f\in L^2(\mathbb{R}^d)$, $\|f\|=1$, and choose the representation of Γ via the 0-net in (2.2):

$$\Gamma(P)=\lim_{N\to\infty}\frac{1}{(2N)^{2d}}\int\limits_{[-N,N]^d}\int\limits_{[-N,N]^d}\big\langle T(-\omega,-t)Px, x\big\rangle\,dt\,d\omega.$$

An easy computation using Plancherel's formula shows that

$$\begin{aligned}\int\limits_{\mathbb{R}^{2d}}\big\langle T(-\omega,-t)Px, x\big\rangle\,dt\,d\omega &= \int\limits_{\mathbb{R}^{2d}}\langle PM_\omega S_t x, M_\omega S_t x\rangle\,dt\,d\omega\\ &= \int\limits_{\mathbb{R}^{2d}}\big|\langle f, M_\omega S_t x\rangle\big|^2\,dt\,d\omega = \int\limits_{\mathbb{R}^d}\Bigg(\int\limits_{\mathbb{R}^d}\Bigg|\int\limits_{\mathbb{R}^d}\overline{f(u)}(S_t x)(u)e^{-2\pi i\omega\cdot u}\,du\Bigg|^2\,d\omega\Bigg)\,dt\\ &= \int\limits_{\mathbb{R}^d}\Bigg(\int\limits_{\mathbb{R}^d}\big|f(u)\big|^2\big|(S_t x)(u)\big|^2\,du\Bigg)\,dt = \int\limits_{\mathbb{R}^d}\big|f(u)\big|^2\Bigg(\int\limits_{\mathbb{R}^d}\big|x(u+t)\big|^2\,dt\Bigg)\,du\\ &= \|f\|^2\|x\|^2<\infty.\end{aligned}$$

Hence, $\Gamma(P)=0$ and, since γ is a faithful state, we conclude that $P=0$. $\square$

Next, we consider certain subalgebras of $\mathfrak{U}$ which we call *causal* following [12].

Definition 5.1. Let $A\in\mathfrak{U}$ and $\mathbb{S}_A\subset\hat{\mathbb{G}}\times\mathbb{G}$ be the smallest semigroup of $\hat{\mathbb{G}}\times\mathbb{G}$ that contains $\Lambda(A)$. The element A is called *causal* if $-\mathbb{S}_A\cap\mathbb{S}_A=\{0\}$ and *hypercausal* if, in addition, $0\notin\Lambda(A)$. We denote the set of all causal and hypercausal elements by $\mathcal{C}$ and $\mathcal{HC}$, respectively. If $\mathbb{S}\subset\hat{\mathbb{G}}\times\mathbb{G}$ is a semigroup with $-\mathbb{S}\cap\mathbb{S}=\{0\}$ then we let $\mathcal{C}(\mathbb{S})=\{B\in\mathcal{C}\colon\ \Lambda(B)\subseteq\mathbb{S}\}$ and $\mathcal{HC}(\mathbb{S})=\{B\in\mathcal{HC}\colon\ \Lambda(B)\subseteq\mathbb{S}\}$.

It is not hard to see [12] that $\mathcal{C}(\mathbb{S})$ is a closed subalgebra of $\mathfrak{U}$ and $\mathcal{HC}(\mathbb{S})$ is a proper two-sided ideal in $\mathcal{C}(\mathbb{S})$. The *causal spectrum* $\sigma_{\mathbb{S}}(A)$ is the spectrum of $A\in\mathcal{C}(\mathbb{S})$ in the Banach algebra $\mathcal{C}(\mathbb{S})$.

Theorem 5.8. *Let $\mathbb{G}$ satisfy Assumption* 5.1 *and $A\in\mathcal{C}$ be such that $(\Lambda(A)-\lambda)\cap(\Lambda(A)\cup\{0\})=\{0\}$ for some $\lambda\in\Lambda(A)$. Then $A^2\neq A$ unless $A\in\{0,I\}$.*

Proof. $A\sim\sum_\lambda c_\lambda U_\lambda\in\mathfrak{U}$ satisfy the assumptions of the theorem and assume for the contrary that $A^2=A\notin\{0,I\}$. From (4.1) we infer that $c_0^2=c_0$, and, hence, either $A\in\mathcal{HC}$ or $I-A\in\mathcal{HC}$. It remains to apply Theorem 4.4 to get a contradiction. $\square$

Remark 5.2. If $\mathbb{S}_A$ is a finitely generated semigroup then the only projections in $\mathcal{C}$ are 0 and I. This can be proved using the technique developed in [12, §8].

Corollary 5.9. *Let $A\in\mathcal{C}$ and assume the semigroup $\mathbb{S}_A$ satisfies at least one of the following conditions*:

(1) *$\mathbb{S}_A$ is a finitely generated semigroup*;
(2) *$\mathbb{S}_A$ satisfies*

$$\big(\mathbb{S}_A\backslash\{0\}\big)+\big(\mathbb{S}_A\backslash\{0\}\big)\neq\mathbb{S}_A\backslash\{0\}.$$

Then the causal spectrum $\sigma_{\mathbb{S}(A)}$ is connected. In particular, any contour in the infinite connected component $\rho_\infty(A)$ of the resolvent set $\rho(A)$ does not separate the spectrum $\sigma(A)$.

Proof. Assume for the contrary that $\sigma_{\mathbb{S}(A)}$ is not connected. Then there exists a non-trivial Riesz projection $P\in\mathcal{C}(\mathbb{S}_A)$ and we get a contradiction with Theorem 5.8 or Remark 5.2. $\square$

The above results are interesting not only in themselves but also in view of the following long-standing conjecture.

HRT Conjecture. *Let $A\in End(L^2(\mathbb{R}^d))$ be a finite linear combination of time-frequency shifts. Then A has no eigen-vectors.*

The conjecture has been proved for many special cases (see in [5]), but the general case, to the best of our knowledge, remains open. Below are a few relevant propositions that can be inferred easily from the above results.

Proposition 5.10. *Let $A \in End(L^2(\mathbb{R}^d))$ be a finite linear combination of time-frequency shifts. Then A has no isolated eigen-values with finite-dimensional eigen-spaces.*

Proof. Follows immediately from Corollary 5.7. □

Proposition 5.11. *If HRT fails, then there is a counterexample $A \in \mathcal{C}$ such that the causal spectrum $\sigma_{\mathbb{S}(A)}$ is connected. In particular, any contour in the infinite connected component $\rho_\infty(A)$ of the resolvent set $\rho(A)$ does not separate the spectrum $\sigma(A)$.*

Proof. Follows immediately from Corollary 5.9. □

Proposition 5.12. *Let $A \in End(L^2(\mathbb{R}^d))$ be a finite linear combination of time-frequency shifts with the rationally independent Bohr spectrum $\Lambda(A)$. Then $\sigma(A)$ is invariant under rotations around 0 in $\mathbb{C}$. In particular, 0 is the only possible isolated point in $\sigma(A)$.*

Proof. Follows immediately from Theorem 4.2. □

Example 5.1. Let $A \in End(L^2(\mathbb{R}))$ be such that

$$\Lambda(A) \subset \{(1,0), (0,1), (\sqrt{2}, \sqrt{2})\}.$$

To the best of our knowledge it is not known if such an operator satisfies HRT. From the above proposition we infer that 0 could be the only isolated eigen-value of A. However, since HRT holds in the lattice case (by Linnell's proof [29]), 0 is not an eigen-value of A. Hence, $\sigma(A)$ has no isolated eigen-values.

Acknowledgments

The authors thank A. Baskakov, K. Gröchenig, and the anonymous referee for their helpful comments. The second author is also grateful for the support of J.J. Benedetto and the other organizers of the February Fourier Talks at the University of Maryland, where some of this research was discussed.

References

[1] E. Acosta-Reyes, A. Aldroubi, I. Krishtal, On stability of sampling-reconstruction models, Adv. Comput. Math. 31 (1–3) (2009) 5–34.
[2] A. Aldroubi, A. Baskakov, I. Krishtal, Slanted matrices, Banach frames, and sampling, J. Funct. Anal. 255 (7) (2008) 1667–1691.
[3] A.B. Antonevič, Invertibility conditions for operators with a convex rationally independent system of shifts, Dokl. Akad. Nauk SSSR 256 (1) (1981) 11–14 (in Russian); translation in: Soviet Math. Dokl. 23 (1) (1981) 1–4.
[4] T.V. Azarnova, Estimates for the elements of the inverse matrices for a class of operators with matrices of special structures, Mat. Zametki 72 (1) (2002) 3–10 (in Russian); translation in: Math. Notes 72 (1–2) (2002) 3–9.
[5] R. Balan, The noncommutative Wiener lemma, linear independence, and spectral properties of the algebra of time-frequency shift operators, Trans. Amer. Math. Soc. 360 (7) (2008) 3921–3941.
[6] R. Balan, P. Casazza, C. Heil, Z. Landau, Density, overcompleteness, and localization of frames. I, II, J. Fourier Anal. Appl. 12 (2) (2006) 105–143; J. Fourier Anal. Appl. 12 (3) (2006) 309–344.
[7] A.G. Baskakov, Spectral criteria for almost periodicity of solutions of functional equations, Mat. Zametki 24 (2) (1978) 195–206; English translation: Math. Notes 24 (1979) 606–612.
[8] A.G. Baskakov, Wiener's theorem and asymptotic estimates for elements of inverse matrices, Funct. Anal. Appl. 24 (1990) 222–224.
[9] A.G. Baskakov, Asymptotic estimates for elements of matrices of inverse operators, and harmonic analysis, Sibirsk. Mat. Zh. 38 (1) (1997) 14–28 (in Russian); translation in: Siberian Math. J. 38 (1) (1997) 10–22.
[10] A.G. Baskakov, Asymptotic estimates for elements of matrices of inverse operators, Izv. Ross. Akad. Nauk Ser. Mat. 61 (6) (1997) 3–26 (in Russian); translation in: Izv. Math. 61 (6) (1997) 1113–1135.
[11] A.G. Baskakov, Representation theory for Banach algebras, Abelian groups, and semigroups in the spectral analysis of linear operators, Sovrem. Mat. Fundam. Napravl. 9 (6) (2004) 3–151 (in Russian); translation in: J. Math. Sci. (N.Y.) 137 (4) (2006) 4885–5036.
[12] A.G. Baskakov, I.A. Krishtal, Harmonic analysis of causal operators and their spectral properties, Izv. Ross. Akad. Nauk Ser. Mat. 69 (3) (2005) 3–54 (in Russian); translation in: Izv. Math. 69 (3) (2005) 439–486.
[13] A.G. Baskakov, I.A. Krishtal, Spectral analysis of operators with the two-point Bohr spectrum, J. Math. Anal. Appl. 308 (2) (2005) 420–439.
[14] S. Bochner, R.S. Phillips, Absolutely convergent Fourier expansions for non-commutative normed rings, Ann. of Math. (2) 43 (1942) 409–418.
[15] F. Futamura, Localizable operators and the construction of localized frames, Proc. Amer. Math. Soc. 137 (12) (2009) 4187–4197.
[16] I. Gohberg, M.A. Kaashoek, H.J. Woerderman, The band method for positive and strictly contractive extension problems: an alternative version and new applications, Integral Equations Operator Theory 12 (3) (1989) 343–382.
[17] C. Heil, J. Ramanathan, P. Topiwala, Linear independence of time-frequency translates, Proc. Amer. Math. Soc. 124 (1996) 2787–2795.
[18] K. Gröchenig, Localization of frames, Banach frames, and the invertibility of the frame operator, J. Fourier Anal. Appl. 10 (2) (2004) 105–132.
[19] K. Gröchenig, A. Klotz, Noncommutative approximation: inverse-closed subalgebras and off-diagonal decay of matrices, arXiv:0904.0386v1 [math.OA], 2 Apr. 2009.
[20] K. Gröchenig, M. Leinert, Symmetry and inverse-closedness of matrix algebras and functional calculus for infinite matrices, Trans. Amer. Math. Soc. 358 (6) (2006) 2695–2711 (electronic).
[21] K. Gröchenig, M. Leinert, Wiener's Lemma for twisted convolution and Gabor frames, J. Amer. Math. Soc. 17 (1) (2003) 1–18.
[22] K. Gröchenig, Z. Rzeszotnik, Banach algebras of pseudodifferential operators and their almost diagonalization, Ann. Inst. Fourier (Grenoble) 58 (7) (2008) 2279–2314.
[23] K. Gröchenig, Z. Rzeszotnik, T. Strohmer, Convergence analysis of the finite section method and Banach algebras of matrices, Integral Equations Operator Theory 162 (2010) 933–946.

[24] K. Gröchenig, T. Strohmer, Pseudodifferential operators on locally compact abelian groups and Sjstrand's symbol class, J. Reine Angew. Math. 613 (2007) 121–146.
[25] S. Jaffard, Propriétés des matrices «bien localisées» près de leur diagonale et quelques applications, Ann. Inst. H. Poincaré Anal. Non Linéaire 7 (5) (1990) 461–476.
[26] I. Krishtal, K. Okoudjou, Invertibility of the Gabor frame operator on the Wiener amalgam space, J. Approx. Theory 153 (2) (2008) 212–224.
[27] V.G. Kurbatov, Algebras of difference and integral operators, Funct. Anal. Appl. 24 (2) (1990) 156–158.
[28] B.M. Levitan, V.V. Žikov, Almost Periodic Functions and Differential Equations, Cambridge University Press, Cambridge, New York, 1982.
[29] P.A. Linnell, Von Neumann algebras and linear independence of translates, Proc. Amer. Math. Soc. 127 (1999) 3269–3277.
[30] L.H. Loomis, An Introduction to Abstract Harmonic Analysis, D. Van Nostrand Company, Inc., Toronto, New York, London, 1953, x+190 pp.
[31] Yu.I. Lyubich, Introduction to the Theory of Banach Representations, Vishcha Shkola, Kharkov, 1985 (in Russian).
[32] V.S. Rabinovich, S. Roch, B. Silbermann, Fredholm theory and finite section method for band-dominated operators, Integral Equations Operator Theory 30 (1998) 452–495.
[33] F. Riesz, B.Sz. Nagy, Functional Analysis, Dover Publications, New York, 1990.
[34] C.E. Shin, Q. Sun, Stability of localized operators, J. Funct. Anal. 256 (8) (2009) 2417–2439.
[35] J. Sjöstrand, Wiener type algebras of pseudodifferential operators, Séminaire sur les Équations aux Dérivées Partielles, 1994–1995, Exp. No. IV, 21 pp., École Polytech., Palaiseau, 1995.
[36] Q. Sun, Frames in spaces with finite rate of innovations, Adv. Comput. Math. 28 (2008) 301–329.
[37] Q. Sun, Local reconstruction for sampling in shift-invariant space, Adv. Comput. Math. 32 (2010) 335–352.
[38] Q. Sun, Wiener's lemma for infinite matrices, Trans. Amer. Math. Soc. 359 (2007) 3099–3123.
[39] Q. Sun, Wiener's lemma for infinite matrices II, Constr. Approx., in press.
[40] Q. Sun, Wiener's lemma for infinite matrices with polynomial off-diagonal decay, C. R. Math. Acad. Sci. Paris 340 (8) (2005) 567–570.
[41] Q. Sun, Wiener's lemma for localized integral operators, Appl. Comput. Harmon. Anal. 25 (2008) 148–167.
[42] M.A. Šubin [Shubin], Almost periodic functions and partial differential operators, Uspekhi Mat. Nauk 33 (2) (1978) 3–47 (in Russian); translation in: Russian Math. Surveys 33 (2) (1978) 1–52.
[43] Nik Weaver, Mathematical Quantization, Stud. Adv. Math., Chapman & Hall/CRC, Boca Raton, FL, 2001, xii+278 pp.
[44] N. Wiener, Tauberian theorems, Ann. of Math. (2) 33 (1) (1932) 1–100.